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AXISYMMETRIC PROBLEMS OF THE THEORY OF ELASTICITY FOR A TRUNCATED HOLLOW CONE†

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An exact solution of the axisymmetric problem of the theory of elasticity for a hollow circular cone, truncated by two spherical surfaces (the ends of the cone), taking into account the natural weight or temperature (the inhomogeneous Lamé equations) is given. The conditions of sliding clamping are satisfied on the conical surfaces, the stresses on one of the ends of the cone are specified, while the boundary conditions can be arbitrary on the other end.

The case of inhomogeneous Lamé equations has also been considered previously for other conditions on the ends [1], but the method used cannot be transferred to the case of inhomogeneous Lamé equations. The non-axisymmetric problem has been considered for a hollow circular cone, including when there is a along the generatrix, and for inhomogeneous Lamé equations [2, 3], the results obtained in [2] need to be refined (see in [4]); other boundary conditions on the ends and on the conical surfaces were considered in [3].

1. FORMULATION OF THE PROBLEM

We consider an elastic body (with shear modulus G and Poisson's ratio μ) in a spherical system of coordinates r, θ, φ, fixed by the relations

$$a_0 \le r \le a_1, \quad \omega_0 \le \theta \le \omega_1, \quad -\pi \le \varphi < \pi \tag{1.1}$$

In the axisymmetric case, the problem of the theory of elasticity for such a body is split into the problem of axisymmetric deformation with the required displacements u_r and u_θ and the problem of twisting with the required displacement u_φ. We will consider axisymmetric deformation. Introducing the notation

$$\begin{gathered} 2Gu_r = u, \quad 2Gu_\theta = \upsilon, \quad \mu_0 = (1-2\mu)^{-1}, \quad \mu_* = \mu_0 + 1, \\ \mu_{**} = \mu_0 + 2 = \kappa\mu_0, \quad \kappa = 3 - 4\mu \end{gathered} \tag{1.2}$$

and denoting a derivative with respect to r by a prime, and a derivative with respect to θ by a dot, we obtain the following system of equations for the functions u and υ (see, for example, [5])

$$\begin{gathered} (r^2u')' - 2u + \frac{1}{\mu_*}\frac{(\sin\theta \dot{u})^{\cdot}}{\sin\theta} - \frac{\mu_{**}}{\mu_*}\frac{(\sin\theta\,\upsilon)^{\cdot}}{\sin\theta} + \frac{\mu_0 r}{\mu_*}\frac{(\sin\theta\,\upsilon')^{\cdot}}{\sin\theta} = -\frac{2r^2}{\mu_*}q_r \\ (r^2\upsilon')' + \mu_*\left|\frac{(\sin\theta\,\dot{\upsilon})^{\cdot}}{\sin\theta} - \frac{\upsilon}{\sin^2\theta}\right| + \mu_0 r\dot{u}' + 2\mu_*\dot{u} = -2r^2q_\theta \end{gathered} \tag{1.3}$$

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Here q_r and q_θ are the components of the intensity of the body forces. The stresses can be expressed in terms of functions introduced by the formulae [5]

$$(1-2\mu)r\sigma_r = (1-\mu)ru' + 2\mu u + \mu\frac{(\sin\theta v)^{\bullet}}{\sin\theta}$$
$$(1-2\mu)r\sigma_\theta = u + \mu ru' + (1-\mu)v^{\bullet} + \mu\,\mathrm{ctg}\,\theta v, \quad 2r\tau_{r\theta} = rv' - v + u^{\bullet} \tag{1.4}$$

We will assume that the conditions of the first fundamental problem

$$\sigma_r(a_1, \theta) = -p_1(\theta), \quad \tau_{r\theta}(a_1, \theta) = q_1(\theta) \tag{1.5}$$

are specified on the spherical surface $r = a_1$.

The following conditions of sliding clamping are satisfied on the conical surfaces $\theta = \omega_i$ ($i = 0, 1$)

$$v(r, \omega_i) = 0, \quad \tau_{r\theta}(r, \omega_i) = 0 \tag{1.6}$$

On the remaining spherical surface $r = a_0$ all possible cases of the boundary conditions can be satisfied: the conditions of the first fundamental problem

$$\sigma_r(a_0, \theta) = -p_0(\theta), \quad \tau_{r\theta}(a_0, \theta) = q_0(\theta) \tag{1.7}$$

the conditions of the second fundamental problem

$$u(a_0, \theta) = 0, \quad v(a_0, \theta) = 0 \tag{1.8}$$

and the conditions of sliding clamping

$$u(a_0, \theta) = 0, \quad \tau_{r\theta}(a_0, \theta) = 0 \tag{1.9}$$

2. THE CASE WHEN THE PROBLEM ALLOWS OF ELEMENTARY SOLUTIONS

We will construct an exact solution [1] of the boundary-value problem for the homogeneous ($q_r = q_\theta = 0$) equations (1.3) in the region (1.1) for the case when the conditions of sliding clamping (1.6) are satisfied, and the conditions of the first fundamental problem (1.5) and (1.6) are satisfied on the spherical surfaces, but with

$$q_i(\theta) = 0, \quad i = 0, 1 \tag{2.1}$$

using the new integral transformation.

However, it has not been noted that the problem allows of a simple solution when

$$p_i(\theta) = p_i \equiv \text{const}, \quad q_i(\theta) = 0, \quad i = 0, 1 \tag{2.2}$$

In fact, if we assume

$$v(r, \theta) \equiv 0, \quad u(r, \theta) = \varphi(r), \quad a_0 \le r \le a_1, \quad \omega_0 \le \theta \le \omega_1 \tag{2.3}$$

the boundary conditions (1.6) will be satisfied, and the homogeneous equations (1.3) will degenerate into a single equation

$$[r^2\varphi'(r)]' - 2\varphi(r) = 0, \quad a_0 < r < a_1 \tag{2.4}$$

It can be verified that the general solution of Eq. (2.4) has the form

$$\varphi(r) = C_0(r) + C_1 r^{-2}; \quad C_i = \text{const}, \quad i = 0, 1 \tag{2.5}$$

Taking relations (2.3), (1.4) and (2.5) into account, we can satisfy boundary conditions (1.5) and (1.6), assuming conditions (2.2). As a result we obtain

$$[C_0, C_1] = \frac{[\tilde{\mu}^{-1}(p_1a_1^3 - p_0a_0^3), a_0^3a_1^3(p_1 - p_0)/2]}{a_1^3 - a_0^3}, \quad \tilde{\mu} = \frac{1+\mu}{1-2\mu}$$

and a field of the displacements and stresses for the problem in question will be given by the formulae

$$\begin{aligned} u(r, \theta) &= C_0r - C_1r^{-2}, \quad v(r, \theta) \equiv 0, \quad \tau_{r\theta}(r, \theta) \equiv 0 \\ [\sigma_r, \sigma_\theta] &= [\tilde{\mu}, (1+\mu)\mu_0]C_0 + [2, 1]C_1r^{-3} \end{aligned} \tag{2.6}$$

We obtain the same elementary solution when there is sliding clamping (1.9) or complete adhesion (1.8) on the spherical boundary $r = a_0$. In both cases the displacement and stress fields will be given by formulae (2.6), and only the formulae for the normal stresses

$$[\sigma_r, \sigma_\theta] = \tilde{\mu}C_0 + [2, 1]C_1r^{-3} \tag{2.7}$$

will be changed and there will be other values for C_0 and C_1

$$[C_0, C_1] = a_1^3p_1(\tilde{\mu}a_1^3 + 2a_0^3)^{-1}[-1, a_0^3] \tag{2.8}$$

The formulae obtained are the same for a hollow cone ($\omega_i \neq 0$, $i = 0, 1$) as for a solid cone ($\omega_1 \neq 0$, $\omega_0 = 0$). They will also be true for a spherical layer $a_0 \leq r \leq a_1$, $0 \leq \theta \leq \pi$, $-\pi \leq \varphi < \pi$. However, the elementary solutions obtained lose their meaning if requirement (2.2) breaks down or Eqs (1.3) themselves turn out to be in homogeneous ($q_r, q_\theta \neq 0$). In that case the method used previously [1] enables one to dispense with requirement (2.1), but it turns out to be inapplicable if Eqs (1.3) are inhomogeneous, i.e. when the problem of free thermoelasticity is solved and, of course, on the right-hand sides of Eqs (1.3) the previously obtained derivatives of the temperature [5] occur.

Equations (1.3) correspond to the case when there are body forces present. If the axis of the cone (1.1) is directed vertically upwards, and we take the force of gravity as the body forces (the specific gravity of the material of the cone is denoted by γ), then in Eqs (1.3) we must take

$$q_r = -\gamma\cos\theta, \quad q_\theta = \gamma\sin\theta, \quad \omega_0 < \theta < \omega_1 \tag{2.9}$$

Below we will describe a method of constructing an exact solution of the above problems for a cone (1.1) in the case of inhomogeneous equations (thermoelasticity), in particular when the natural weight is taken into account, i.e. when relations (2.9) are satisfied.

3. REDUCTION OF THE PROBLEMS TO A VECTOR ONE-DIMENSIONAL BOUNDARY-VALUE PROBLEM

We will apply an integral transformation with respect to the variable θ to the system of equations (1.3), so that boundary conditions (1.5) are satisfied, and which are equivalent to the conditions

$$v(r, \omega_i) = 0, \quad u^\bullet(r, \omega_i) = 0; \quad i = 0, 1 \tag{3.1}$$

In order that these conditions should be satisfied, it is necessary to apply the previously obtained integral transformation [1, 6] to Eqs (1.3), i.e. to change to the transforms

$$\begin{aligned} v_k(r) &= \int_{\omega_0}^{\omega_1} \varphi_a^1(\theta, \nu_k)v(r, \theta)\sin\theta d\theta = \int_{\omega_0}^{\omega_1} y(\theta, \nu_k)v(r, \theta)\sin\theta d\theta \\ u_k(r) &= \int_{\omega_0}^{\omega_1} \varphi_c^0(\theta, \nu_k)u(r, \theta)\sin\theta d\theta = \int_{\omega_0}^{\omega_1} y_*(\theta, \nu_k)u(r, \theta)\sin\theta d\theta; \quad k = 0, 1, 2, \ldots \end{aligned} \tag{3.2}$$

The second equalities are written in terms of the previously employed notation [6]

$$y(\theta, \nu_k) = y_0(\theta, \nu_k)|_{\mu=1} = \varphi_a^1(\theta, \nu_k)$$
$$y_*(\theta, \nu_k) = y_1(\theta, \nu_k)|_{\mu=0,\,h_0=0,\,h_1=0} = \varphi_c^0(\theta, \nu_k) \tag{3.3}$$

(here μ is the superscript in the spherical functions $P_\nu^\mu(\cos\theta)$ and $Q_\nu^\mu(\cos\theta)$ [7], which satisfy Legendre's equation and were used previously in [6], where, in order to write the boundary conditions, the real numbers h_0 and h_1 were introduced, and for the conditions used here $h_0 = h_1 = 0$). In this case [1, 6]

$$y(\theta, \nu) = P_\nu^1(\cos\theta)Q_\nu^1(\cos\omega_1) - P_\nu^1(\cos\omega_1)Q_\nu^1(\cos\theta), \quad \nu = \nu_k$$
$$y_*(\theta, \nu) = P_\nu(\cos\theta)Q_\nu^1(\cos\omega_1) - P_\nu^1(\cos\omega_1)Q_\nu(\cos\theta), \quad \nu = \nu_k \tag{3.4}$$

$$y_*^{\cdot}(\theta, \nu_k) = y(\theta, \nu_k) \tag{3.5}$$

and ν_k $(k = 0, 1, 2, \ldots)$ are the roots of the transcendental equation

$$\Omega_\nu^1 \equiv P_\nu^1(\cos\omega_0)Q_\nu^1(\cos\omega_1) - P_\nu^1(\cos\omega_1)Q_\nu^1(\cos\omega_0) = 0 \tag{3.6}$$

The eigenfunctions (3.4) satisfy Legendre's equation when $\mu = 1$ and $\mu = 0$ respectively (in the latter case the superscript $\mu = 0$ is omitted). The follow conditions are satisfied for these

$$y(\omega_i, \nu_k) = 0, \quad y_*^{\cdot}(\omega_i, \nu_k) = 0, \quad i = 0, 1 \tag{3.7}$$

The following inversion formulae have been established [1, 6] for transforms (3.2)

$$\upsilon(r, \theta) = -\sum_{k=0}^{\infty} \upsilon_k(r)\frac{2\nu_k + 1}{\nu_k(\nu_k + 1)}\left[S_\nu^1\frac{\partial\Omega_\nu^1}{\partial\nu}\right]_{\nu=\nu_k}^{-1} y(\theta, \nu_k)$$
$$u(r, \theta) = -\sum_{k=0}^{\infty} u_k(r)(2\nu_k + 1)\left[S_\nu^1\frac{\partial\Omega_\nu^1}{\partial\nu}\right]_{\nu=\nu_k}^{-1} y_*(\theta, \nu_k) \tag{3.8}$$

$$S_\nu^1 = P_\nu^1(\cos\omega_1)/P_\nu^1(\cos\omega_0) = Q_\nu^1(\cos\omega_1)/Q_\nu^1(\cos\omega_0), \quad \nu = \nu_k, \quad k = 0, 1, \ldots$$

In order to change to the transforms (3.2) in Eqs (1.3) we must multiply the first equation of (1.3) by $\sin\theta y_*(\theta, \nu_k)$ and the second by $\sin\theta y(\theta, \nu_k)$ and integrate by parts over the section $[\omega_0, \omega_1]$. As a result we will have

$$[r^2u_k'(r)]' - 2u_k(r) - \mu_*^{-1}[Nu_k(r) - \mu_{**}\upsilon_k(r) + \mu_0 r\upsilon_k'(r)] = r^2q_k$$
$$[r^2\upsilon_k'(r)]' - \mu_*N\upsilon_k(r) + 2\mu_*Nu_k(r) + \mu_0Nru_k'(r) = -r^2q_k^*, \quad a_0 < r < a_1 \tag{3.9}$$

where

$$N = \nu_k(\nu_k + 1), \qquad \left\|\begin{matrix} q_k \\ q_k^* \end{matrix}\right\| = \gamma\int_{\omega_0}^{\omega_1}\left\|\begin{matrix} \sin 2\theta y(\theta, \nu_k)\mu_*^{-1} \\ 2\sin^2\theta y_*(\theta, \nu_k) \end{matrix}\right\| d\theta \tag{3.10}$$

Here we must take into account Legendre's equation which the eigenfunctions $y(\theta, \nu_k)$ and $y^*(\theta, \nu_k)$ satisfy, and also relations (3.1), (3.7), (3.5) and (2.9).

Boundary conditions (1.5) and (1.8) in transforms (3.2) can be written in the form

$$a_1\upsilon_k'(a_1) - \upsilon_k(a_1) + Nu_k(a_1) = 2a_1q_{1k}$$
$$(1 - \mu)a_1u_k'(a_1) - 2\mu u_k(a_1) - \mu\upsilon_k(a_1) = -(1 - 2\mu)a_1p_{1k} \tag{3.11}$$

$$u_k(a_0) = 0, \quad v_k(a_0) = 0 \tag{3.12}$$

where

$$p_{ik} = \int_{\omega_0}^{\omega_1} p_i(\theta) y_*(\theta, \nu_k) \sin\theta d\theta, \quad q_{ik} = \int_{\omega_0}^{\omega_1} q_i(\theta) y(\theta, \nu_k) \sin\theta d\theta; \quad i = 0, 1 \tag{3.13}$$

Boundary conditions (1.7) in transforms (3.2) are converted into conditions (3.11) with a_1 replaced by a_0, while boundary conditions (1.9) are written in the form

$$u_k(a_0) = 0, \quad a_0 v_k'(a_0) - v_k(a_0) = 0 \tag{3.14}$$

Hence, all the versions of the problem can be reduced to a one-dimensional boundary-value problem for Eqs (3.9). It is more convenient to transfer this boundary-value problem, specified in the section $[a_0, a_1]$, to the section $[\alpha, 1]$, $\alpha = a_0/a_1 < 1$ using the replacement

$$r = a_1\rho, \quad u_k(a_1\rho) = \tilde{u}_k(\rho), \quad v_k(a_1\rho) = \tilde{v}_k(\rho) \tag{3.15}$$

For the new required functions, the system of equations (3.9) retains its previous form (if we denote a derivative with respect to ρ by a prime) and will be specified in the section $\alpha < \rho < 1$.

Boundary conditions (3.11), (3.12) and (3.14) must also be written for the new functions $\tilde{u}_k(\rho)$, $\tilde{v}_k(\rho)$. For example, boundary conditions (3.11) and (3.12) take the form

$$\begin{aligned} &\tilde{v}_k'(1) - \tilde{v}_k(1) + N u_k(1) = 2a_1 q_{1k} \\ &(1-\mu)\tilde{u}_k'(1) - 2\mu\tilde{u}(1) - \mu\tilde{v}_k(1) = -(1-2\mu)a_1 p_{1k} \\ &\tilde{u}_k(\alpha) = 0, \quad \tilde{v}_k(\alpha) = 0 \end{aligned} \tag{3.16}$$

We will write the one-dimensional boundary-value problem obtained in vector form, for which we introduce the required vector $\mathbf{y}(\rho)$ and the specified vector $\mathbf{f}(\rho)$ and matrices of the form

$$\begin{aligned} &\mathbf{y}(\rho) = \left\| \begin{matrix} \tilde{u}_k(\rho) \\ \tilde{v}_k(\rho) \end{matrix} \right\|, \quad \mathbf{f}(\rho) = a_1^2 \left\| \begin{matrix} q_k \\ -q_k^* \end{matrix} \right\| \rho^2 \\ &P = \left\| \begin{matrix} -2 - N\mu_*^{-1} & \mu_{**}\mu_*^{-1} \\ 2\mu_* N & (-\mu_* N) \end{matrix} \right\|, \quad I = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \quad Q = \left\| \begin{matrix} 0 & -\mu_*^{-1} \\ N & 0 \end{matrix} \right\| \end{aligned} \tag{3.17}$$

and also the boundary functionals

$$U_i[\mathbf{y}(\rho)] = A_i \mathbf{y}(\alpha_i) + B_i \mathbf{y}'(\alpha_i), \quad i = 0, 1; \quad \alpha_0 = \alpha, \quad \alpha_1 = 1 \tag{3.18}$$

with matrices and vectors of the form

$$A_1 = \left\| \begin{matrix} N & -1 \\ 2\mu & -\mu \end{matrix} \right\|, \quad B_1 = \left\| \begin{matrix} 0 & 1 \\ 1-\mu & 0 \end{matrix} \right\|, \quad \gamma_1 = \left\| \begin{matrix} -2q_{1k} \\ (1-2\mu)p_{1k} \end{matrix} \right\|, \quad A_0 = I, \; B_0 = 0, \; \gamma_0 = 0 \tag{3.19}$$

We can now write boundary-value problem (3.9), (3.16) in vector form as

$$\begin{aligned} &L_2\mathbf{y}(\rho) \equiv I[\rho^2\mathbf{y}'(\rho)]' + \mu_0 Q\rho\mathbf{y}'(\rho) + P\mathbf{y}(\rho) = \mathbf{f}(\rho), \quad \alpha < \rho < 1 \\ &U_i[\mathbf{y}(\rho)] \equiv a_i\gamma_i, \quad i = 0, 1 \end{aligned} \tag{3.20}$$

All the versions of this problem can be reduced to the same vector problem (3.20), and only the form of the boundary functionals (3.20) and the vectors γ_0 and γ_1 will be changed.

Similar problems of uncoupled thermoelasticity can be reduced to the same vector problem, and only the form of the vector $\mathbf{f}(\rho)$ will be changed.

4. A PARTICULAR SOLUTION OF THE VECTOR PROBLEM, WHICH GIVES AN EXACT SOLUTION FOR THE SPECIAL CASE OF THE PROBLEMS IN QUESTION

If we obtain Green's matrix $G(\rho, \xi)$ and the basis system of the matrices $\Psi_j(\rho)$ ($j = 0, 1$), the solution of boundary-value problem (3.20) can be written in the form [8]

$$\mathbf{y}(\rho) = \int_\alpha^1 G(\rho, \xi)\mathbf{f}(\xi)d\xi - \Psi_0(\rho)a_0\gamma_0 - \Psi_1(\rho)a_1\gamma_1 \tag{4.1}$$

In order to construct the matrices $G(\rho, \xi)$ and $\Psi_j(\rho)$, we need to know the fundamental system $Y_j(\rho)$ ($j = 0, 1$), i.e. to construct two linearly independent solutions of the matrix equation

$$L_2Y(\rho) = 0, \quad \alpha < \rho < 1 \tag{4.2}$$

It can be shown that the solution of Eq. (4.2) will be the function

$$Y(\rho) = \frac{1}{2\pi i}\oint_C \rho^s M^{-1}(s)ds \tag{4.3}$$

Here $M^{-1}(s)$ is the inverse of the matrix

$$M(s) = \mathbf{I}s(s+1) + \mu_0 Qs + P = \begin{Vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{Vmatrix}$$

$$m_{11} = s(s+1) - 2 - \mu_*^{-1}N, \quad m_{12} = -(\mu_0 s - \mu_{**})\mu_*^{-1}, \tag{4.4}$$
$$m_{21} = (\mu_0 s + 2\mu_*)N, \quad m_{22} = s(s+1) - \mu_* N$$

obtained by carrying out the operations $L_2\rho^s\mathbf{I}$, and C is a closed contour enveloping the pole of the matrix $M^{-1}(s)$.

By the well-known scheme for constructing an inverse matrix, we obtain

$$M^{-1}(s) = \frac{1}{p_4(s)}\begin{Vmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{Vmatrix}$$
$$p_4(s) = \det M(s) = s^4 + 2s^3 - (2N+1)s^2 - 2(N+1)s + N(N-2) = \prod_{j=1}^4 (s - s_j) \tag{4.5}$$

where

$$s_1 = \nu_k + 1, \quad s_2 = \nu_k - 1, \quad s_3 = -\nu_k, \quad s_4 = -\nu_k - 2 \tag{4.6}$$

It can be seen that the integral (4.3) is equal to the sum of the residues at the poles (4.6). Then, if we assume $\nu_k > 1$ ($k = 0, 1, 2, \ldots$), the residues at the first two poles give functions which increase as $\rho \to \infty$, while at the remaining poles they decrease as $\rho \to \infty$.

In order to evaluate the matrix integral (4.3), it is sufficient, by Eq. (4.5), to have available the values of the following scalar integrals

$$[\varphi(\rho), \varphi_1(\rho), \varphi_2(\rho)] = \frac{1}{2\pi i}\oint_C \frac{\rho^s[1, s, s(s+1)]}{p_4(s)}ds \tag{4.7}$$

It is then only necessary to obtain $\varphi(\rho)$, since it can be shown that

$$\varphi_1(\rho) = \rho\varphi'(\rho), \quad \varphi_2(\rho) = [\rho^2\varphi'(\rho)]'$$

Evaluating the integrals (4.7) as the sums of the residues at the poles $s_1 = \nu + 1, s_2 = \nu - 1$, we obtain the solutions $Y_0(\rho)$ of matrix equations (4.2), which increases as $\rho \to \infty$. If we take the residues at the poles $s_3 = -\nu, s_4 = -\nu - 2$, we obtain the second solution $Y_1(\rho)$ of Eq. (4.2), which decreases at infinity and is linearly independent with respect to $Y_0(\rho)$.

Carrying out these operations we obtain the formulae

$$Y_0(\rho) = \rho^{\nu+1}R_{\nu+1}A_+(\nu) - \rho^{\nu-1}R_\nu B_+(\nu), \quad Y_1(\rho) = \rho^{-\nu}R_\nu A_-(\nu) - \rho^{-\nu-2}R_{\nu+1}B_-(\nu)$$
$$R_\nu = [2(2\nu+1)(2\nu-1)]^{-1} \tag{4.8}$$

The matrices $A_\pm(\nu)$ and $B_\pm(\nu)$ are defined by the formulae ($\mu_1 = [2(1-\mu)]^{-1}$)

$$A_+(\nu) = \begin{Vmatrix} 2(\nu+1) - \mu_0 N & \mu_*^{-1}(\mu_0\nu - 2) \\ -\mu_0 N(\nu+\kappa+2) & \mu_1 N + 2\nu \end{Vmatrix}$$

$$B_+(\nu) = \begin{Vmatrix} -(\mu_0 N + 2\nu) & 2 - \mu_1\nu \\ \mu_0 N(\nu+\kappa) & -[2(\nu+1) - \mu_1 N] \end{Vmatrix}$$

$$A_-(\nu) = \begin{Vmatrix} -\mu_0 N - 2\nu & -\mu_1(\nu+\kappa) \\ \mu_0 N[\nu - 4(1-\mu)] & \mu_1 N - 2(\nu+1) \end{Vmatrix} \tag{4.9}$$

$$B_-(\nu) = \begin{Vmatrix} -[\mu_0 N - 2(\nu+1)] & -\mu_1(\nu+\kappa+2) \\ N(\mu_0\nu - 2) & \mu_1 N + 2\nu \end{Vmatrix}$$

In formulae (4.8) and (4.9) and everywhere below the symbol ν means the eigenvalues $\nu_k (k = 0, 1, 2, \ldots)$.

Hence, the general solution of Eq. (4.2) will have the form

$$Y(\rho) = Y_0(\rho)C_0 + Y_1(\rho)C_1 \tag{4.10}$$

where C_0 and C_1 are matrix-constants.

If we have the fundamental system of solutions (4.10) of matrix equation (4.2), we can construct a basis system of matrix solutions $\Psi_0(\rho)$ and $\Psi_1(\rho)$, which satisfy the matrix boundary-value problem (δ_{ij} is the Kronecker delta)

$$L_2\Psi_j(\rho) = 0, \quad \alpha < \rho < 1, \quad U_i[\Psi_j] = \delta_{ij}I; \quad i, j = 0, 1 \tag{4.11}$$

By relation (4.10) they will be constructed in the form

$$\Psi_j(\rho) = Y_0(\rho)C_0^{(j)} + Y_1(\rho)C_1^{(j)}, \quad j = 0, 1 \tag{4.12}$$

while the matrices $C_i^{(j)}$ $(i, j = 0, 1)$ are found from boundary conditions (4.11). For example, we will have the following equations for $\Psi_1(\rho)$

$$U_0[Y_0(\rho)]C_0^{(1)} + U_0[Y_1(\rho)]C_1^{(1)} = 0, \quad U_1[Y_0(\rho)]C_0^{(1)} + U_1[Y_1(\rho)]C_1^{(1)} = I \tag{4.13}$$

By relations (3.18), (3.19) and (4.8) we obtain

$$U_0[Y_0(\rho)] = Y_0(\alpha) = \alpha^{\nu-1}R_\nu C_+(\nu), \quad U_0[Y_1(\rho)] = Y_1(\alpha) = \alpha^{-\nu-2}R_\nu C_-(\nu) \tag{4.14}$$

where

$$C_+(\nu) = B_+(\nu) + \alpha^2 b_\nu A_+(\nu), \quad C_-(\nu) = \alpha^2 A_-(\nu) - b_\nu B_-(\nu); \quad b_\nu = R_\nu^{-1} R_{\nu+1} \tag{4.15}$$

Substituting expressions (4.14) into the first equation of (4.13), we obtain the relation

$$C_0^{(1)} = \alpha^{-2\nu-1} D(\nu) C_1^{(1)}, \quad D(\nu) = C_+^{-1}(\nu) C_-(\nu) \tag{4.16}$$

We can similarly obtain

$$\begin{aligned} U_1[Y_0(\rho)] &= R_{\nu+1}\tilde{A}_+(\nu) + R_\nu \tilde{B}_+(\nu) = D_+(\nu) \\ U_1[Y_1(\rho)] &= R_\nu \tilde{A}_-(\nu) - R_{\nu+1}\tilde{B}_-(\nu) = D_-(\nu) \end{aligned} \tag{4.17}$$

where

$$\tilde{A}_+(\nu) = \Lambda(\nu) A_+(\nu), \quad \tilde{B}_+(\nu) = \Lambda(\nu - 2) B_+(\nu)$$

$$\tilde{A}_-(\nu) = \Lambda(-\nu - 1) A_-(\nu), \quad \tilde{B}_-(\nu) = \Lambda(-\nu - 3) B_-(\nu) \tag{4.18}$$

$$\Lambda(\nu) = \begin{Vmatrix} N & \nu \\ \nu + 1 - \mu(\nu - 1) & -\mu \end{Vmatrix}$$

and in all the elements of the matrix $\Lambda(\nu)$ we have made these replacements of the number ν, except $N = \nu(\nu + 1)$, the expression for which remains unchanged.

Substituting expression (4.16) into the second equation of (4.13), after multiplying the equation obtained by $\alpha^{2\nu+1}$, we obtain

$$C_1^{(1)} = \alpha^{2\nu+1}[D_+(\nu)D(\nu) + \alpha^{2\nu+1} D_-(\nu)]^{-1} \tag{4.19}$$

Taking (4.16) into account, we obtain the second matrix constant

$$C_0^{(1)} = D(\nu)[D_+(\nu)D(\nu) + \alpha^{2\nu+1} D_-(\nu)]^{-1} \tag{4.20}$$

Substituting expressions (4.19) and (4.20) into (4.12), we obtain $\Psi_1(\rho)$.

Using similar operations we can also obtain $\Psi_0(\rho)$, i.e. the construction of the basis system of the matrix boundary-value problem (4.1) is completed.

We will consider a special case of the problems of the theory of elasticity formulated above: we ignore the natural weight of the cone (1.1) ($\mathbf{f}_i(\rho) \equiv 0, i = 0, 1$) and assume that boundary conditions (1.5)–(1.7) and (1.9) are satisfied, i.e. $\gamma_0 = 0$, while γ_1 is defined by the third formula of (3.19). The solution of vector problem (3.20), by virtue of relations (4.1), can then be written in the form

$$\begin{Vmatrix} \tilde{u}_k(\rho) \\ \tilde{\upsilon}_k(\rho) \end{Vmatrix} = a_1 \Psi_1(\rho) \begin{Vmatrix} -2q_{1k} \\ (1 - 2\mu)p_{1k} \end{Vmatrix} \tag{4.21}$$

i.e. it is sufficient merely to know the single matrix $\Psi_1(\rho)$, obtained above. From $\tilde{u}_k(\rho)$, $\tilde{\upsilon}_k(\rho)$ obtained, by virtue of formulae (3.15) we obtain the transforms of the displacements

$$u_k(r) = \tilde{u}_k(r/a_1), \quad \upsilon_k(r) = \tilde{\upsilon}_k(r/a_1) \tag{4.22}$$

Substituting these into expressions (3.8) we obtain the displacements themselves and we thereby complete the construction of the exact solution for the special case of the problem under discussion.

We will consider once again the case when $a_0 = 0$, i.e. there is a spike in the cone (1.1). We must then take the limit as $\alpha > 0$ in the formulae obtained. Then

$$C_+(\nu) = B_+(\nu), \quad C_-(\nu) = -b_\nu B_-(\nu) \tag{4.23}$$

and inversion of the matrix $C_+(\nu)$ reduces to inversion of the matrix $B_+(\nu)$. Carrying out this inversion using the well-known procedure, we obtain

$$B_+^{-1}(\nu) = \frac{-\tilde{B}(\nu)}{\mu_0 N h(\nu)}, \quad \tilde{B}(\nu) = \left\| \begin{matrix} 2(\nu+1)-\mu_1 N & \mu_1\nu-2 \\ -\mu_0 N(\nu+\kappa) & \mu_0 N+2\nu \end{matrix} \right\| \tag{4.24}$$

$$h(\nu) = \mu_1 N + 2(\kappa+\mu_1\nu) + (\nu+\kappa)(\mu_1\nu-2)$$

By virtue of relations (4.15), (4.16) and (4.24) the matrix $D(\nu)$ takes the form

$$D(\nu) = b_\nu[\mu_0 N h(\nu)]^{-1}\tilde{B}(\nu)B_-(\nu)$$

and we obtain from relations (4.16) and (4.19)

$$C_0^{(1)} = D(\nu)[D_+(\nu)D(\nu)]^{-1}, \quad C_1^{(1)} = \alpha^{2\nu+1}[D_+(\nu)D(\nu)]^{-1} \tag{4.25}$$

When $\alpha \to 0$, by virtue of expression (4.19) $C_1^{(1)} = 0$, and hence formula (4.12) takes the form

$$\Psi_1(\rho) = Y_0(\rho)D(\nu)[D_+(\nu)D(\nu)]^{-1}$$

As above, knowing $\Psi_1(\rho)$, by Eq. (4.21) we obtain the transforms (4.22), and from them, using the inversion formulae (3.8), we also obtain the required displacements $u(r, \theta)$ and $v(r, \theta)$.

5. CONSTRUCTION OF GREEN'S MATRIX OF THE VECTOR BOUNDARY-VALUE PROBLEM AND THE EXACT SOLUTION OF THE PROBLEM

As we can see, for the complete solution of the vector problem in accordance with expression (4.1) we need to construct Green's matrix $G(\rho, \xi)$.

To do this we first construct the fundamental matrix $\Phi(\rho, \xi)$, i.e. such a matrix that the solution of the inhomogeneous differential equation of the problem (3.20) can be written in the form

$$\mathbf{y}(\rho) = \int_\alpha^1 \Phi(\rho, \xi)\mathbf{f}(\xi)d\xi \tag{5.1}$$

With this aim, we will write the equation mentioned in the form

$$L_2\mathbf{y}(\rho) = \mathbf{f}(\rho), \quad 0<\rho<\infty \tag{5.2}$$

assuming that the right-hand side is only non-zero in the section $[\alpha, 1]$. To find the solution of Eq. (5.2) we will apply an integral Mellin transformation to it, i.e. we change to the transforms

$$\mathbf{y}_s = \int_0^\infty \rho^{s-1}\mathbf{y}(\rho)d\rho, \quad \mathbf{f}_s = \int_\alpha^1 \xi^{-s}\mathbf{f}(\xi)d\xi \tag{5.3}$$

As a result we arrive at the following algebraic equation

$$\tilde{M}(s)\mathbf{y}_s = \mathbf{f}_s, \quad \tilde{M}(s) = M(-s) \tag{5.4}$$

The matrix $M(s)$ is defined by formulae (4.4).

Solving algebraic equation (5.4) and inverting the Mellin transforms (5.3), we obtain a solution of Eq. (5.2) and thereby obtain the fundamental matrix

$$\Phi(\rho, \xi) = \frac{1}{\xi}\Phi\left(\frac{\rho}{\xi}\right), \quad \Phi(x) = \frac{1}{2\pi i}\int_l M^{-1}(-s)x^{-s}ds \tag{5.5}$$

If we assume $\nu_k > 1$ ($k = 0, 1, 2, \ldots$), the contour l will be a straight line parallel to the imaginary axis and intersecting the real axis in the interval (0, 1). Then, as in Section 4, the matrix integral (5.5) reduces to calculating the scalar integrals

$$[\tilde{I}_0(x), \tilde{I}_1(x), \tilde{I}_2(x)] = \frac{1}{2\pi i}\int_l \frac{x^{-s}[1, s, s(s-1)]}{p_4(-s)}ds; \quad \tilde{I}_1(x) = -x\tilde{I}_0'(x), \quad \tilde{I}_2(x) = [x^2\tilde{I}_0'(x)]' \tag{5.6}$$

where it is sufficient merely to evaluate the first integral, while the remaining ones, as previously, can be evaluated from the last two formulae of (4.6). Using the standard scheme of contour integration, we obtain

$$\tilde{I}_0(x) = \frac{1}{2}\begin{cases} x^{-\nu}R_\nu - x^{-(\nu+2)}R_{\nu+1}, & x > 1 \\ x^{\nu-1}R_\nu - x^{\nu+1}R_{\nu+1}, & x < 1 \end{cases} \tag{5.7}$$

Hence, from Eqs (5.5) and (5.6) we have

$$\Phi(x) = \left\| \begin{matrix} \tilde{I}_2(x) - \mu_* N\tilde{I}_0(x) & -\mu_1[\tilde{I}_1(x) + \kappa\tilde{I}_0(x)] \\ N[\mu_0\tilde{I}_1(x) - 2\mu_*\tilde{I}_0(x)] & \tilde{I}_2(x) - (2 + \mu_*^{-1}N)\tilde{I}_0(x) \end{matrix} \right\| \tag{5.8}$$

Green's matrix can now be obtained from the formula [8]

$$G(\rho, \xi) = \Phi(\rho, \xi) - \sum_{j=0}^{1} \Psi_j(\rho)(U_j[\Phi(\rho, \xi)])^{-1} \tag{5.9}$$

The matrices $\Psi_j(\rho)$ are defined by formulae (4.12), where the matrix $\Psi_1(\rho)$ is completely defined and $\Psi_0(\rho)$ is found in a similar way.

Hence, formula (4.1) when $\gamma_0 = 0$, taking expressions (5.9) and (4.12) into account, gives the solution of the vector boundary-value problem, and together with this also an exact solution of the problem of the theory of elasticity in question in the case of boundary conditions (1.5), (1.6) and (1.8). For the boundary conditions (1.7) and (1.9) the proposed scheme is completely preserved; only the form of the boundary functional $U_0[\mathbf{y}(\rho)]$, defined by formula (3.18), is changed.

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